Maximizing the Guarded Boundary of an Art Gallery is APX complete

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2nd ACAC, August 2007

Outline

Motivation
- Art gallery theorems and algorithms
- Applications

A gap preserving reduction
- Promise problems and gap preserving reductions
- Construction part of the reduction
- Transformation of a feasible solution

Conclusion
- Summary
- Open problems

Outline

Visibility in simple polygons

- Two objects are *visible* to each other, if they can be joined by a line segment that avoids all obstacles between the objects.
- Inside a simple polygon the obstacles are the polygon's edges.
- If a point is visible to a guard, we say that the guard *covers* the point.
- The problem is to place guards (point, vertex, edge) that fulfill a covering requirement (boundary, interior).

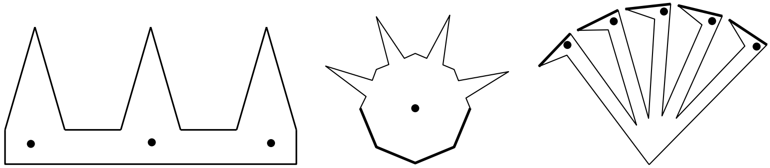
Combinatorial results

Theorem (V. Chvátal, 1975, S. Fisk, 1978)

$\lfloor \frac{n}{3} \rfloor$ *point guards are always sufficient and sometimes necessary in order to cover completely any simple polygon.*

Conjecture (T. Shermer, 1994)

Are $\lfloor \frac{n}{4} \rfloor$ *edge guards always sufficient?*



Algorithmic results

Minimization problems

- NP-hard (D. Lee, A. Lin, 1986)
- $O(\log n)$-approximable (S. Ghosh, 1987)
- NP-hard for 3-link polygons (B. Nilsson, 1995)
- APX-hard (S. Eidenbenz, 1998)
- $\log n$-hard for polygons with holes (S. Eidenbenz, 2000)
- APX-hard for 2-link polygons (B. Brodén et al, 2001)
- 12-approximable for monotone polygons (B. Nilsson, 2005)

Algorithmic results

Maximization problems

- 1 guard: FPTAS (S. Ntafos, M. Tsoukalas, 1994)
- k guards: $O(1)$ deterministically approximable
  (I. Emiris, C. Fragoudakis, E. Markou, S. Zachos, 2002–2006)
  - vertex, edge guards,
  - oversee, watch,
  - length, value, area.
- k guards: $O(1)$ approximable with high probability
  (O. Cheong et al, 2004)

Outline

Applications

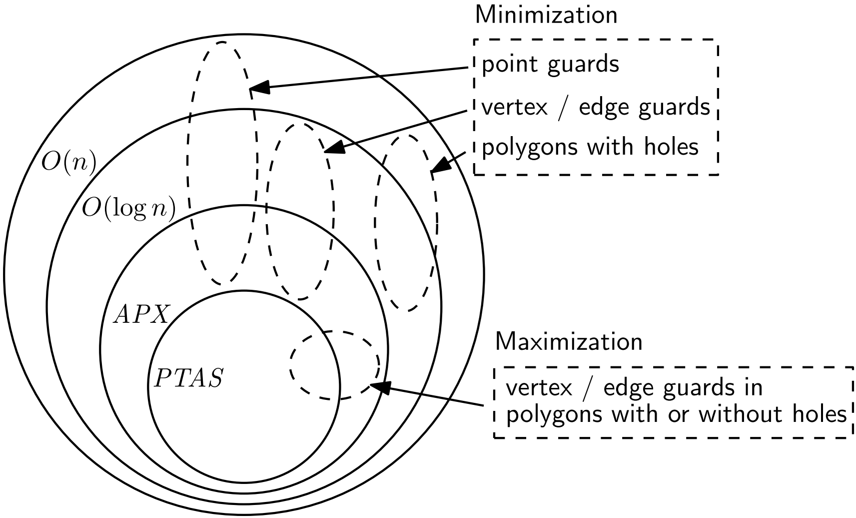
Minimization problems

- Guard exhibits in a gallery using a minimum number of guards.
- Wireless communication technology: place a minimum number of stations so that any point can communicate with at least one station.

Maximization problems

- More realistic setting.
- The given budget allows only a small number of stations.
- Place k stations in a way that most points can communicate.

A hierarchy of approximation classes



Outline

The basic problem that we studied

Definition (Maximum Length Vertex Guard)

Given is a simple polygon P and k vertex guards. The problem asks to place the guards so that the total length of the overseen part of P's boundary from all guards is maximized.

A variation of satisfiability

Definition (Maximum 5 Occurence 3 SAT)

Let Φ be a boolean formula given in CNF, with each clause consisting of at most 3 literals and each variable appearing in at most 5 clauses. The goal is to find a truth assignment for the variables of Φ such that the number of satisfied clauses is maximum.

Theorem (S. Arora, 1997)

For a set of instances of Max-5Occ-3SAT with m clauses an oracle promises that either $OPT(I) = m$ or $OPT(I) < (1 - \epsilon)m$. It is NP-hard to decide which is the case for the optimal solution.

Gap preserving reduction

Definition

It is a polynomial reduction of an instance I of Max-5Occ-3SAT to an instance I' of Max-Length-Vertex-Guard such that:

$$\begin{array}{lcl} OPT(I) = m & \rightarrow & OPT(I') = L(\partial P) \\ OPT(I) < (1-\epsilon)m & \rightarrow & OPT(I') < L(\partial P) - \epsilon m 8 L(e_{short}) \end{array}$$

Lemma

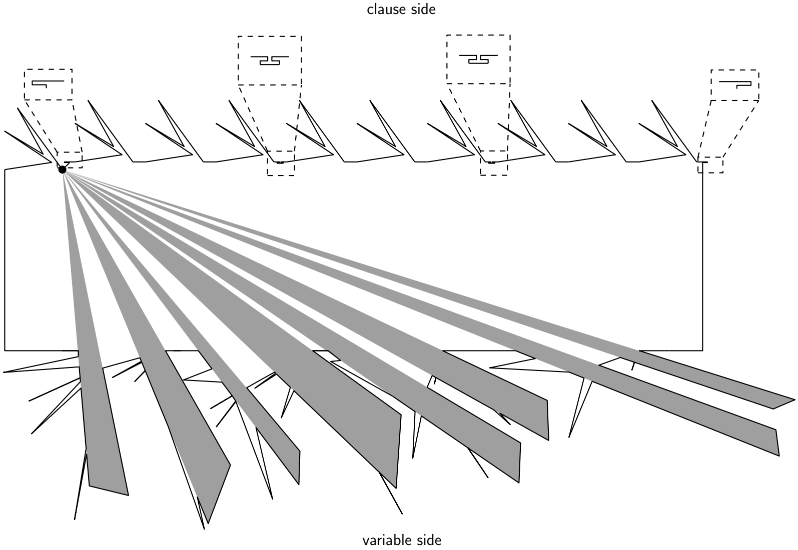
Unless $P = NP$ there is no approximation algorithm for Max-Length-Vertex-Guards that achieves:

$$\frac{OPT(I')}{SOL(I')} < \frac{L(\partial P)}{L(\partial P) - \epsilon m 8 L(e_{short})}$$

Outline

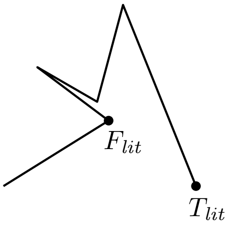
The final polygon

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



The literal pattern

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



The clause pattern

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$

The clause pattern

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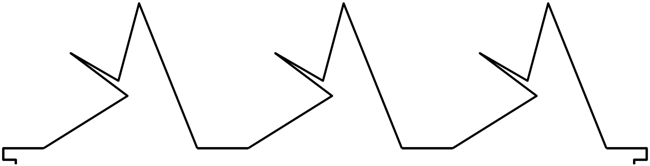

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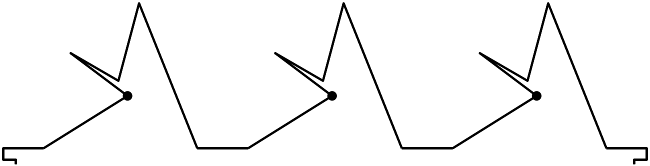
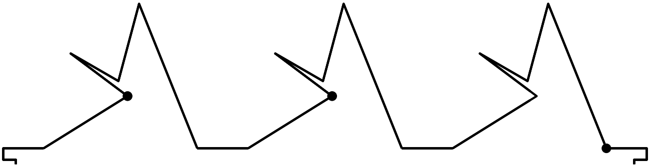


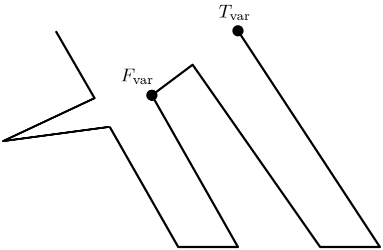
The clause pattern

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



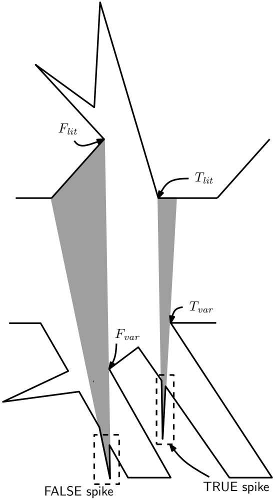
The variable pattern

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



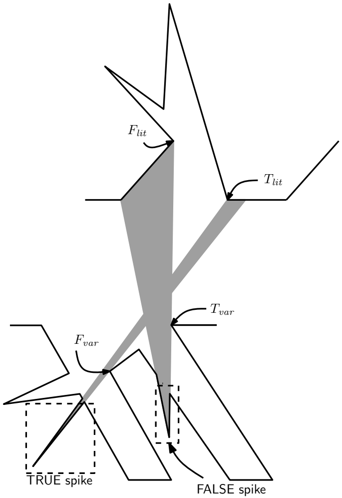
Spikes for a negative literal

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



Spikes for a positive literal

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



Outline

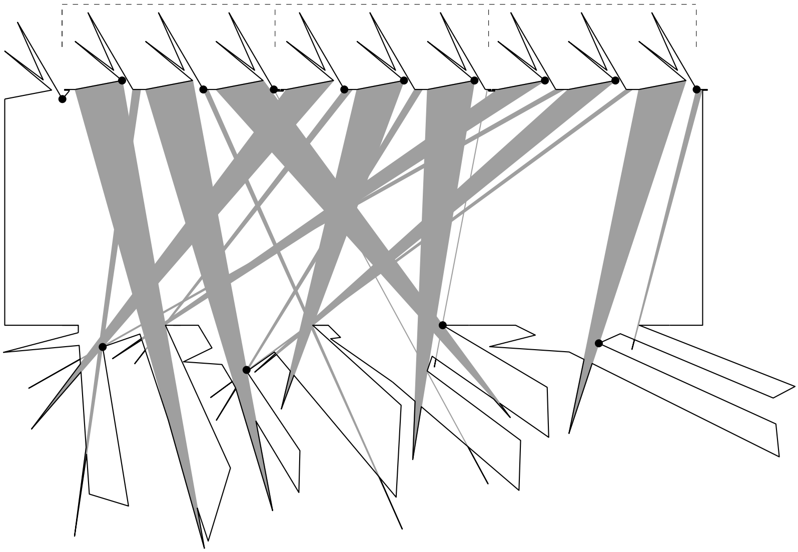
Transformation of a feasible solution

From the truth assignment to the guard placement

- Suppose that a truth assignment satisfies all clauses of Φ. We place vertex $k = |lit| + |var| + 1$ guards in the polygon:
  - 1 guard on the distinguished vertex,
  - 1 guard on every variable pattern,
  - 1 guard on every literal pattern.
- If some clauses are not satisfied then the "short" edges in the corresponding clause patterns are not covered.
- Generally, the whole polygon is covered except possibly some "short" edges.

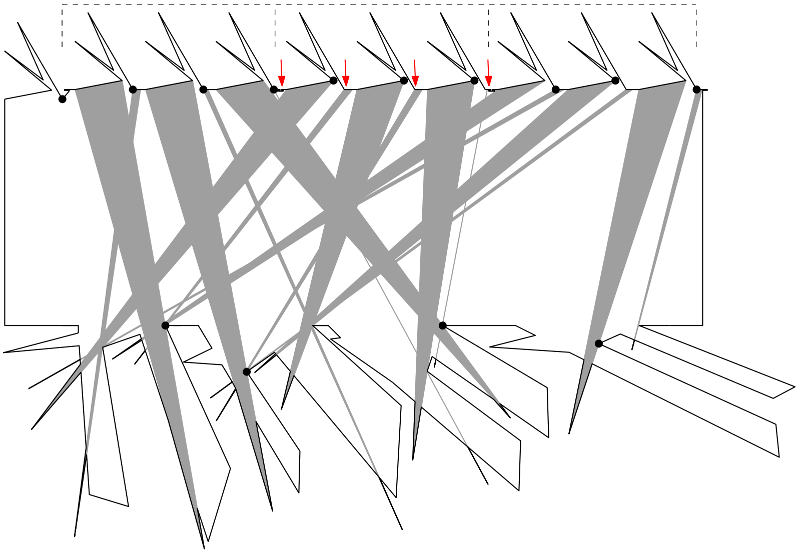
The whole polygon is covered

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



Some "short" edges are not covered

$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_4)$



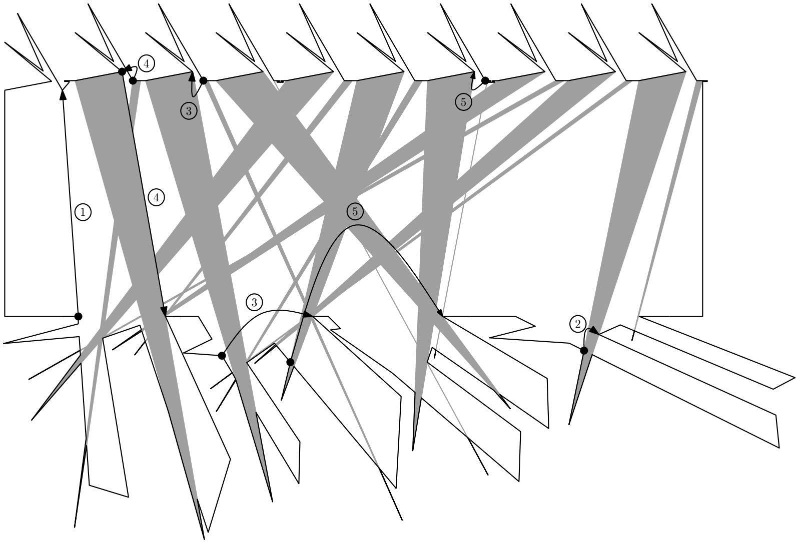
Transformation of a feasible solution

From the guard placement to the truth assignment

Given a guard placement, construct a modified guard placement:

- move guards so that the whole polygon (possibly except a number of “short” edges) is guarded,
- guarded length is at least as before
- consistent guard placement at F_{lit}, T_{lit}, F_{var}, T_{var}.

Moving the given guards



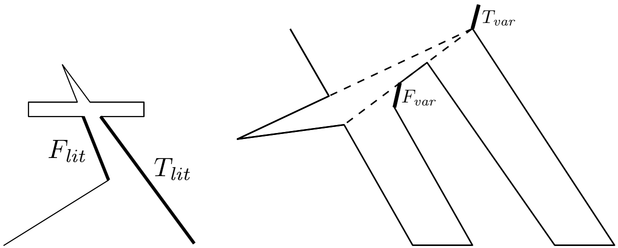
Inapproximability result

Instances I' of Maximum Length Vertex Guard

Let $\epsilon' = \frac{8\epsilon L(e_{short})}{3L_v+3L_l+6L_s+L_c+L_r}$, then there is no polynomial time approximation algorithm that achieves:

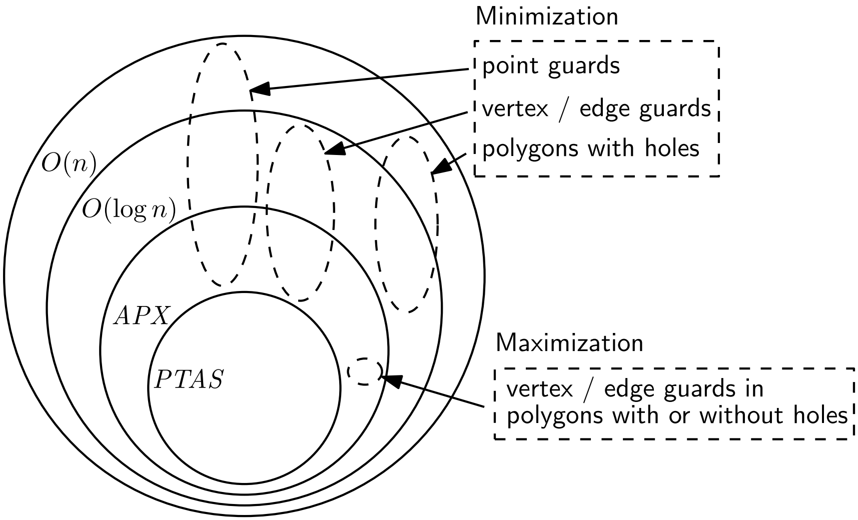
$$\frac{OPT(I')}{SOL(I')} < 1 + \frac{\epsilon'}{1-\epsilon'}$$

Patterns for edge guards



A hierarchy of approximation classes

Final placement of maximization problems



Outline

Summary

- We investigated the problem of maximizing the guarded boundary of an art gallery.
- We used vertex or edge guards.
- The same reasoning applies for several maximization variations (value, area, etc.)
- We proved that, unless P=NP all these variations do not admit a PTAS.
- Our main contribution is a gap preserving reduction from Max-5Occ-3SAT to Max-Length-Vertex Guards.

Outline

Open problems

- An approximation algorithm for Minimum-Point-Guards with any guarantee of performance better than $O(n)$.
- Approximation algorithms for the maximization problems using point guards (inside the polygon).
- Robust implementations in order to gain intuition.

Thank you! Any questions?