Budgeted Coverage of a Maximum Part of a Polygonal Area

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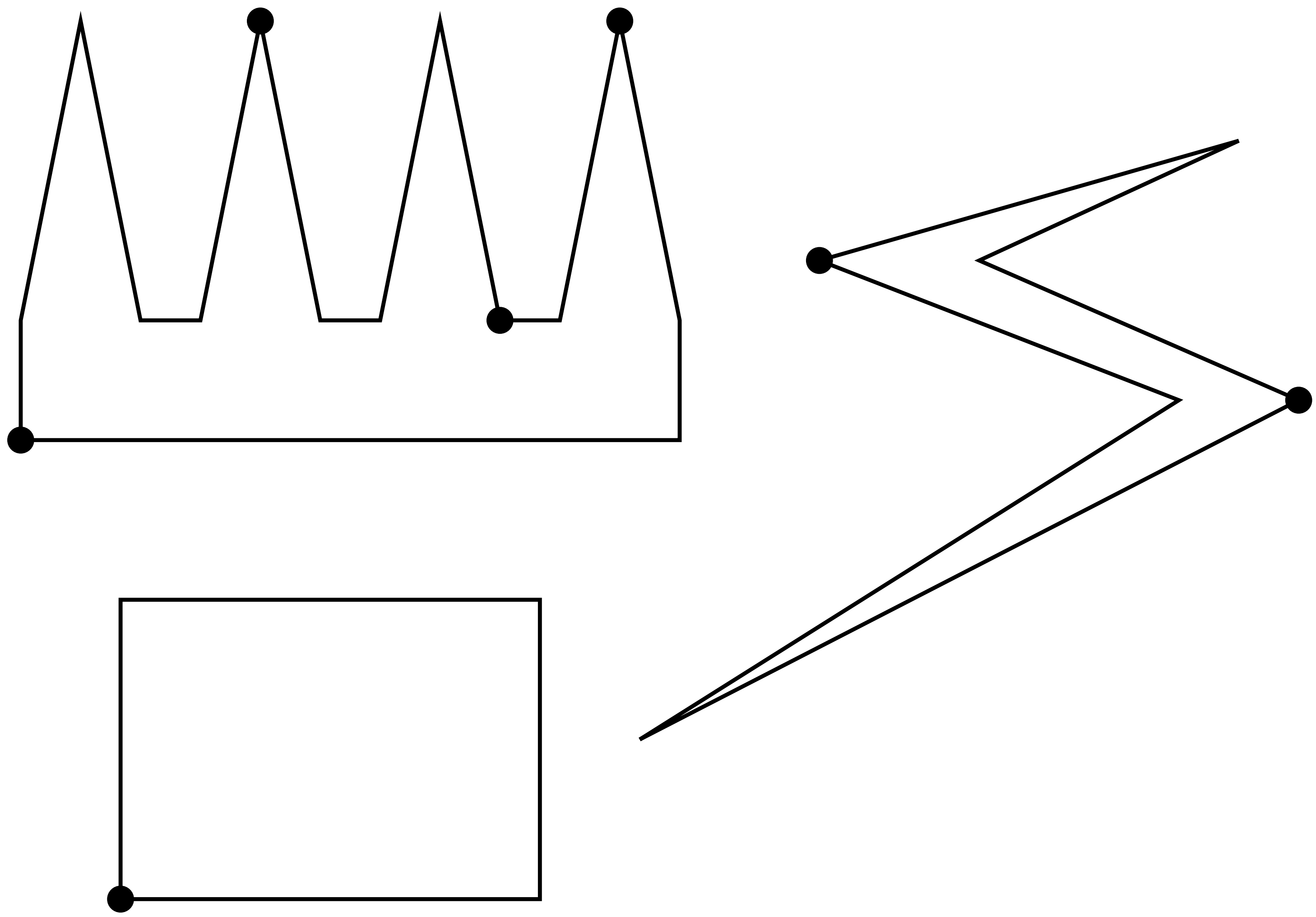
A classical Covering Problem

- **Art Gallery Problem**

  MINIMUM VERTEX GUARD problem (NP-hard)

- In terms of **Wireless Communication Technology**:
  - Imagine a polygonal geographical area,
  - The area is considered **covered** if every point of the area can communicate with at least one station,
  - The goal is to cover the area by placing a minimum number of stations.

Classical Examples



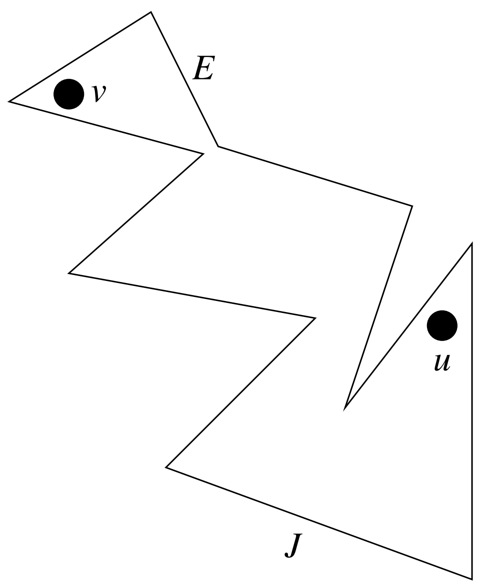
Model Properties

- Communication networks use such high frequency ranges that side effects of reflection and refraction become important unless the two points are mutually visible. Thus a straight line of sight approach, models reality with sufficient precision (points on a straight line segment).
- Two points can communicate directly or through a station.
- Communication (visibility) between two points is possible or blocked, depending on the area topology.
- Graphs, terrains and polygons with or without holes have been used as models.

Restricted Models

- A **Visibility Graph** is accosiated with a polygon: Its vertex set is the vertex set of the polygon and two vertices share an edge in the visibility graph iff they are mutually visible in the polygon.
- Difficult problems in general graphs are easy in Visibility Graphs (e.g. Maximum Clique).
- **Polygons with holes** can represent quite general topologies: for every graph, we can construct a polygon with holes such that two vertices are mutually visible iff they share an edge in the graph.

Visibility Predicates



Guards v and u are not mutually visible, guard v **oversees** edge E and guard u **watches** edge J:

- $\neg sees(v, u) : \exists x \in \overline{vu} : x \notin P$
- $oversees(v, E) : \forall x \in E : sees(v, x)$
- $watches(u, J) : \exists J' \subseteq J : oversees(u, J')$

Finest Visibility Segmentation (FVS)

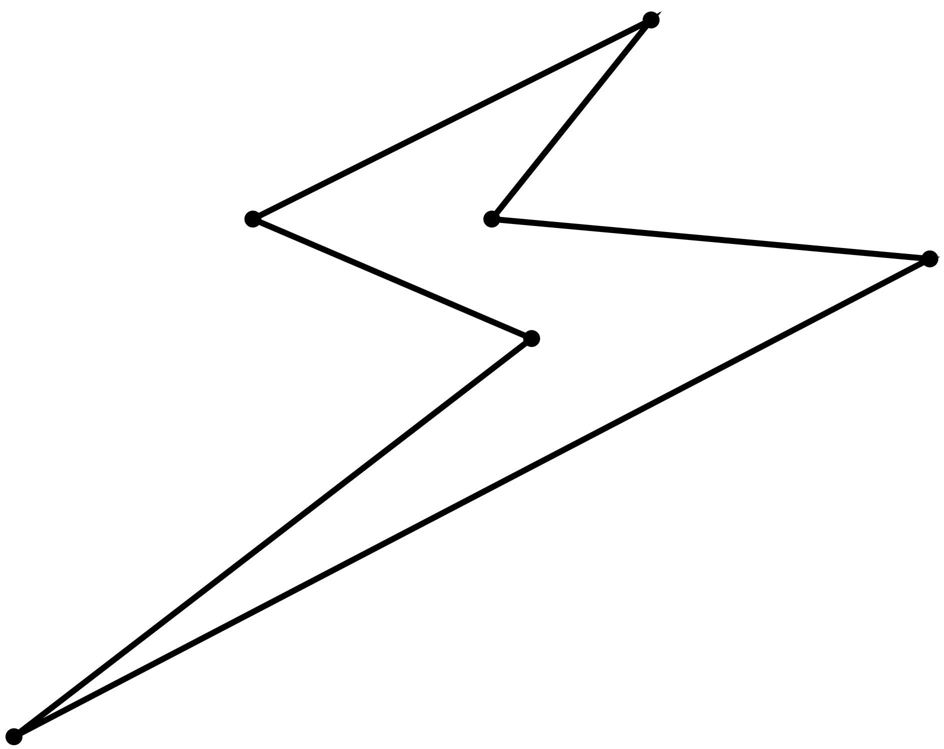
Discretize the boundary of the polygon with respect to visibility:

- Use the Visibility Graph $V_G(P)$ of the polygon,
- Extend edges of $V_G(P)$ inside polygon P up to the boundary of P

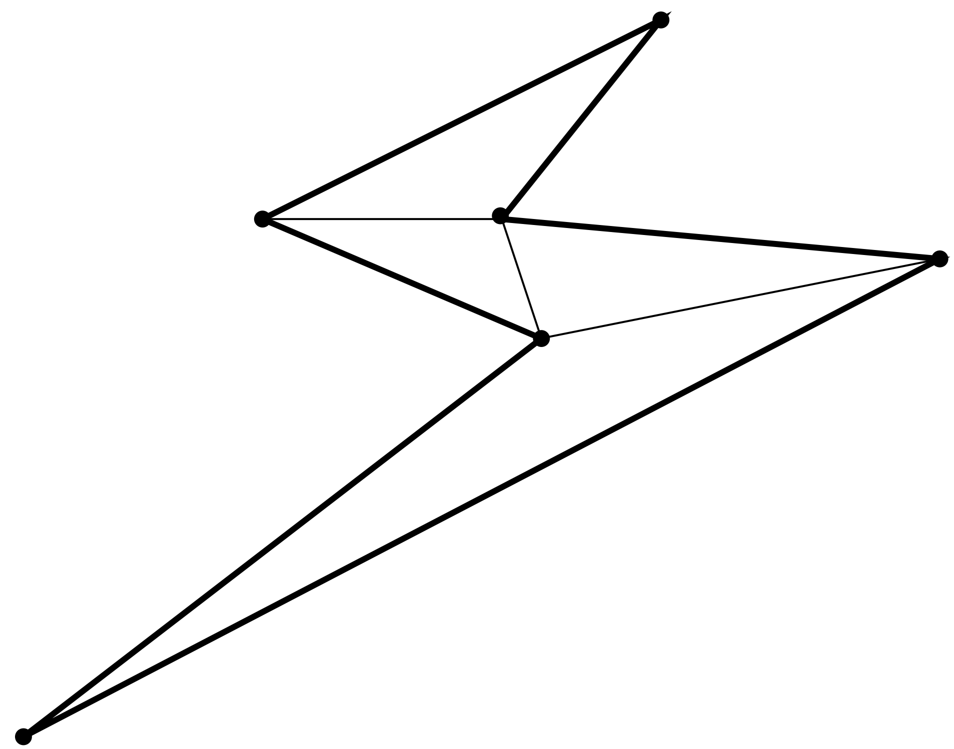
We obtain $O(n^2)$ points which are endpoints of line segments with the following property:

> *For any vertex $v \in V(P)$, a segment (a, b) defined by consecutive FVS points a, b is* overseen *by v iff it is* watched *by v.*

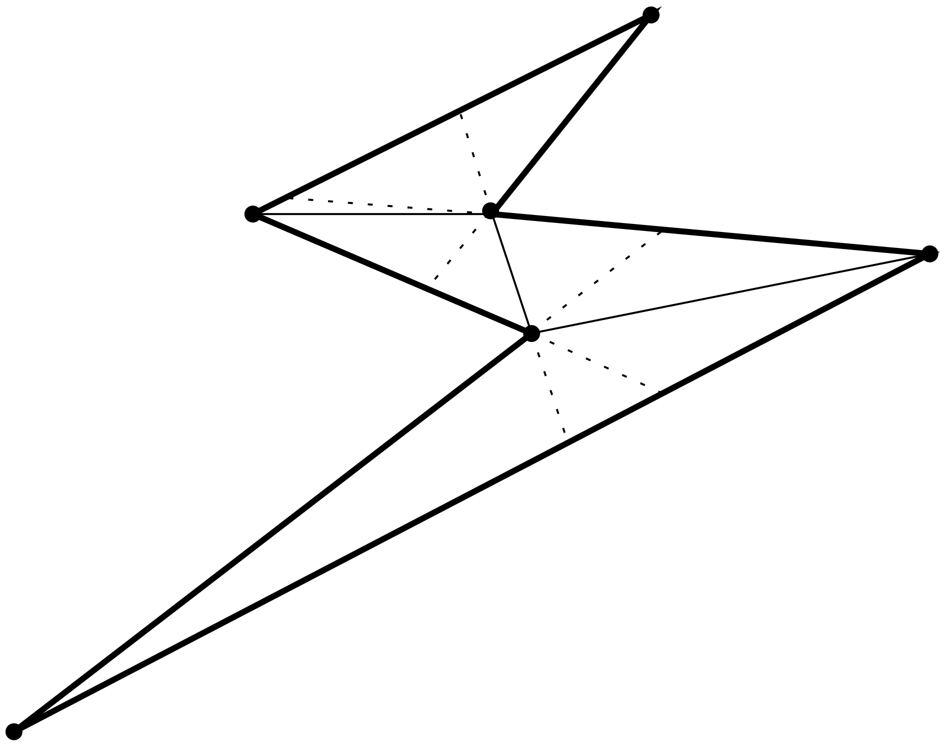
A polygon's FVS



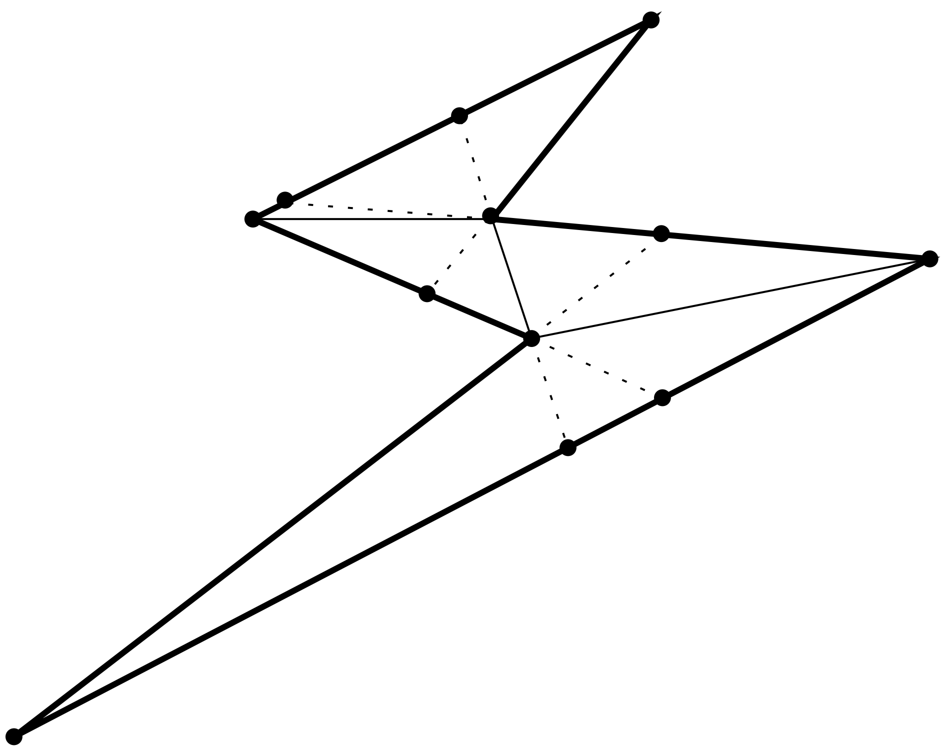
A polygon's FVS



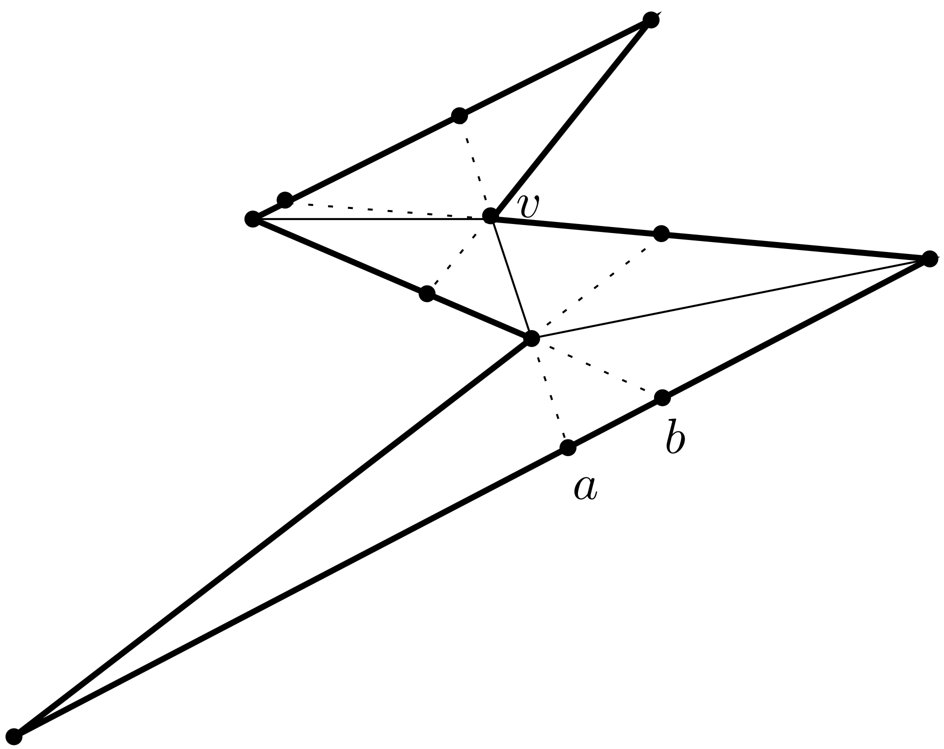
A polygon's FVS



A polygon's FVS



A polygon's FVS



The Budgeted Maximum Length Vertex Guard Problem

Definition: Given is a polygon P with costs on vertices and a number $B > 0$. Let $L(b)$ be the euclidean length of the line segment b. The goal of the Budgeted Maximum Length Vertex Guard (BMLVG) problem is to place vertex guards so that the euclidean length of that part of P's boundary that is **overseen** by the guards is **maximum** and the total cost of vertices with guards is at most B.

The BMLVG problem (cont.)

Fact: BMLVG problem is NP-hard.

Proof: MAXIMUM LENGTH VERTEX GUARD (MLVG) problem is a special case of BMLVG problem where each vertex of the polygon has cost 1. The decision version of MINIMUM VERTEX GUARD problem reduces to the corresponding decision version of MLVG problem.

Approximation Algorithm (sketch)

- For every $v \in V(P)$ construct a set $E'(v)$ of line segments on P's boundary that are visible from vertex v (use the FVS construction).

- In every step of the greedy algorithm which maximizes the overall overseen boundary, place a guard on a vertex so that a maximum increase in the $\frac{length}{cost}$ ratio is achieved. The cost of the augmented guard collection G should not exceed budget B.

- The solution is $L(G)$ unless there exists a vertex v that oversees more length on the boundary than G. If $L(v) > L(G)$ the solution is $L(v)$.

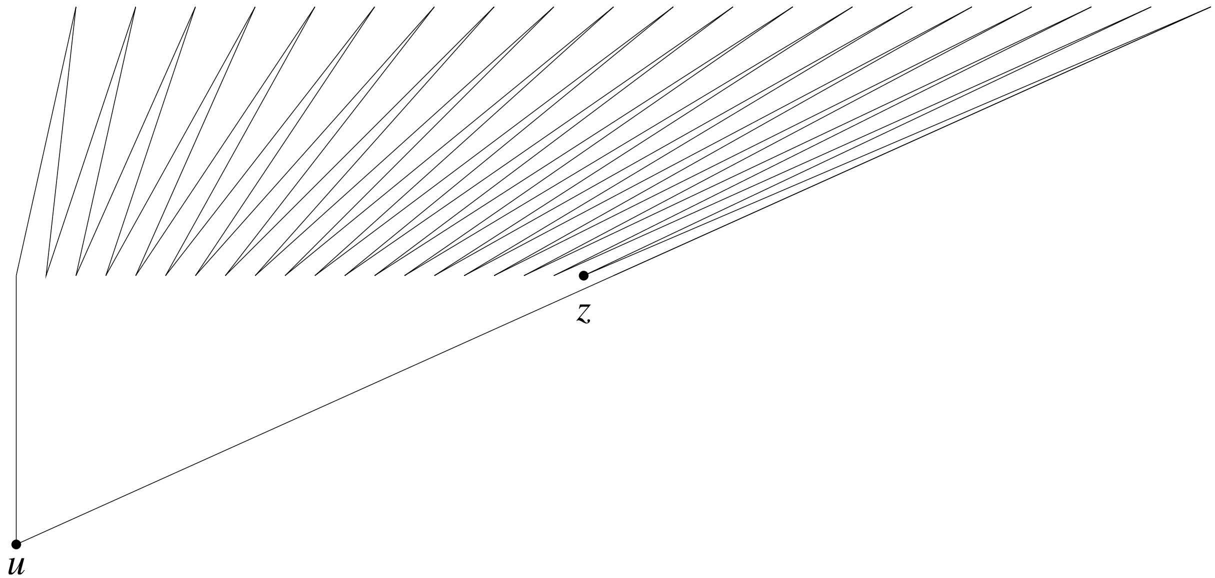
Approximation Algorithm (details)

$SOL \leftarrow \emptyset,\ G \leftarrow \emptyset,\ C_{TOT} \leftarrow 0,\ M \leftarrow V$
repeat
  select $v_i \in V$ that maximizes $\frac{L(G \cup E'(v_i))}{c_i}$
  if $C_{TOT} + c_i \leq B$ **then**
    $G \leftarrow G \cup E'(v_i)$
    $C_{TOT} \leftarrow C_{TOT} + c_i$
  end if
  $M \leftarrow M - \{v_i\}$
until $M = \emptyset$
select $v_t \in V$ that maximizes $L(E'(v_t))$
if $L(G) \geq L(E'(v_t))$ **then**
  $SOL \leftarrow G$
else
  $SOL \leftarrow E'(v_t)$
end if
return $L(SOL)$

Finally it holds:

$L(SOL) \geq 0,316\ L(OPT)$

Example of a special case



$$\forall v \in V(P) \setminus \{z\} : c(v) = 162,\ c(z) = 35$$
$$B = 162$$

$$\forall v \in V(P) \setminus \{u, z\} : L(u) > L(z) > L(v)$$
$$L(OPT) = 161 = L(u)$$
$$L(z) = 35$$

The BUDGETED MAXIMUM VALUE VERTEX GUARD Problem

Definition: Given is a polygon P and an integer $B > 0$. Assume the boundary of P is subdivided into disjoint line segments with non negative weights. Every vertex has a cost value. The goal of the BUDGETED MAXIMUM VALUE VERTEX GUARD Problem (BMVVG) is to place vertex guards so that the total weight of the **watched** set of line segments is **maximum** and the total cost of vertices with guards is at most B.

The BMVVG problem (cont.)

Fact: BMVVG problem is NP-hard.

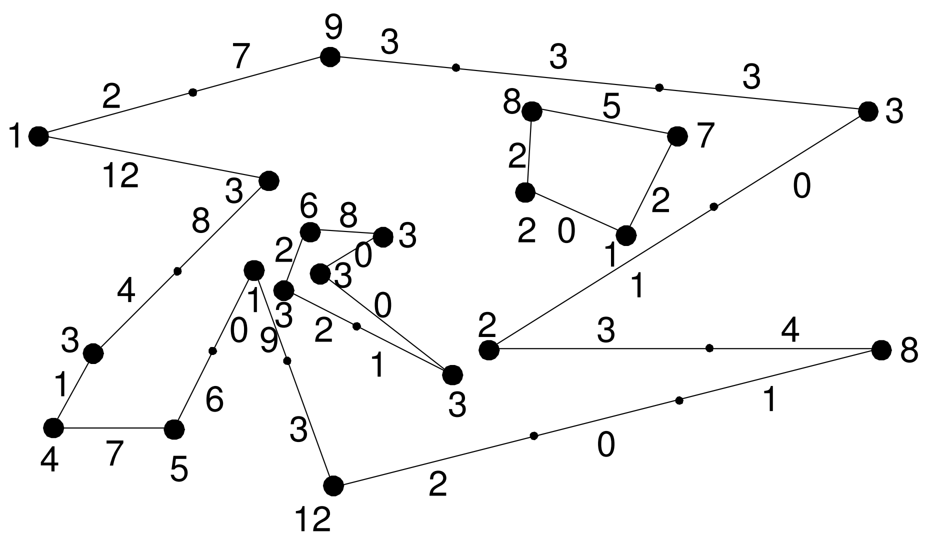
Proof: Maximum Value Vertex Guard (MVVG) problem is a special case of BMVVG problem where each vertex of the polygon has cost 1. The decision version of Minimum Vertex Guard problem reduces to the corresponding decision version of MVVG problem.

Application

The polygon models the geographical space. An interpretation of weighted line segments are **inhabited areas**. The weight interpretation is the **population** of an area. Imagine a number of towns lying on the boundary of a polygonal geographical area (perhaps a lake).

The goal is to place a number of stations such that the total number of people that can communicate is maximized. The total cost of the stations must be within a budget B.

A Weighted Polygon with holes and Costs on vertices



Approximation Algorithm (sketch)

- For every $v \in V(P)$ construct a set $E'(v)$ of line segments on P's boundary that are watched by v.
- In every step of the greedy algorithm which maximizes the overall watched value, place a guard on a vertex so that a maximum increase in the $\frac{weight}{cost}$ ratio is achieved. The cost of the augmented guard collection G should not exceed budget B.
- The solution is $W(G)$ unless there exists a vertex v that oversees more length on the boundary than G. If $W(v) > W(G)$ the solution is $W(v)$.

Approximation Algorithm (details)

$SOL \leftarrow \emptyset$, $G \leftarrow \emptyset$, $C_{TOT} \leftarrow 0$, $M \leftarrow V$
repeat
  select $v_i \in V$ that maximizes $\frac{W(G \cup E'(v_i))}{c_i}$
  if $C_{TOT} + c_i \leq B$ **then**
    $G \leftarrow G \cup E'(v_i)$
    $C_{TOT} \leftarrow C_{TOT} + c_i$
  end if
  $M \leftarrow M - v_i$
until $M = \emptyset$
select $v_t \in V$ that maximizes $W(E'(v_t))$
if $W(G) \geq W(E'(v_t))$ **then**
  $SOL \leftarrow G$
else
  $SOL \leftarrow E'(v_t)$
end if
return $W(SOL)$

Finally it holds:

$$W(SOL) \geq 0,316W(OPT)$$

More Variations

- Edge Guards
- Overseeing instead of watching

Conclusion

We investigated the following problems for polygons with or without holes:

- Budgeted Maximum Length Vertex/ Edge Guard
- watching Budgeted Maximum Value Vertex / Edge Guard
- overseeing Budgeted Maximum Value Vertex / Edge Guard

All the above problems are NP-hard and we provided polynomial time approximation algorithms with constant ratio using a greedy technique based on the approximation of Budgeted Maximum Coverage problem.

Conclusion (cont.)

While investigating the above problems we used:

- **weights** on pieces of the polygon's boundary,
- the useful and promising concept of **watching** a set as opposed to completely **overseeing** it,
- we introduced **costs** on candidate guard places.