How to Place Efficiently Guards and Paintings in an Art Gallery

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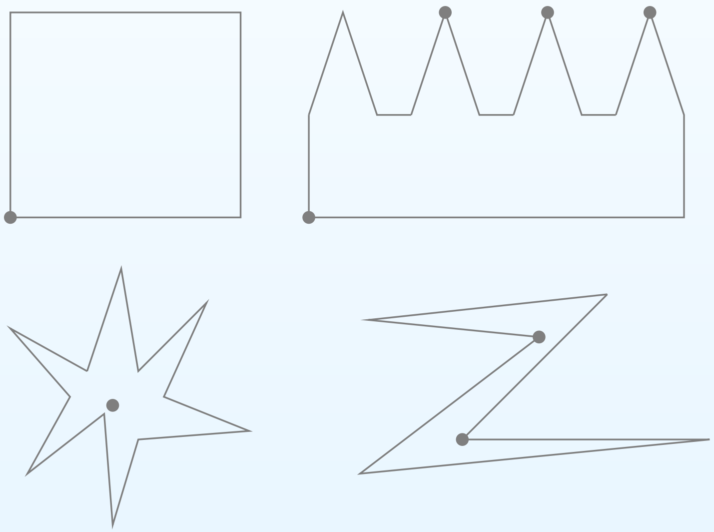
Outline

Introduction

The Art Gallery (AG) Problem

A polygon is given and the goal is to place as few as possible guards in the polygon, so that the polygon is covered.



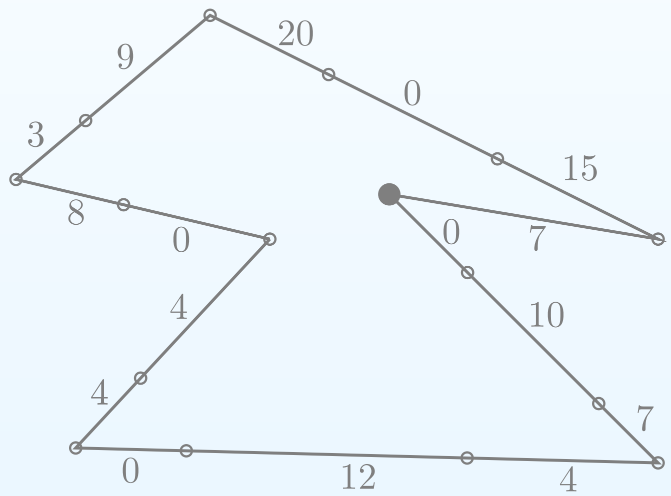
In a variation a number of guards is given and the goal is to cover as many parts of the polygon as possible.

Variations of the AG Problem

- Polygons with holes.
- The points that must be covered lie:
  - on the boundary of the polygon and of its holes,
  - inside the polygon.
- The guards can be placed:
  - on vertices,
  - inside edges,
  - generaly inside the polygon.
- The guards are realized as:
  - vertices (vertex guards),
  - whole edges (edge guards).

Our Problem

How to place exhibits, like paintings, and guards in an art gallery so that the total value of guarded paintings is a maximum?

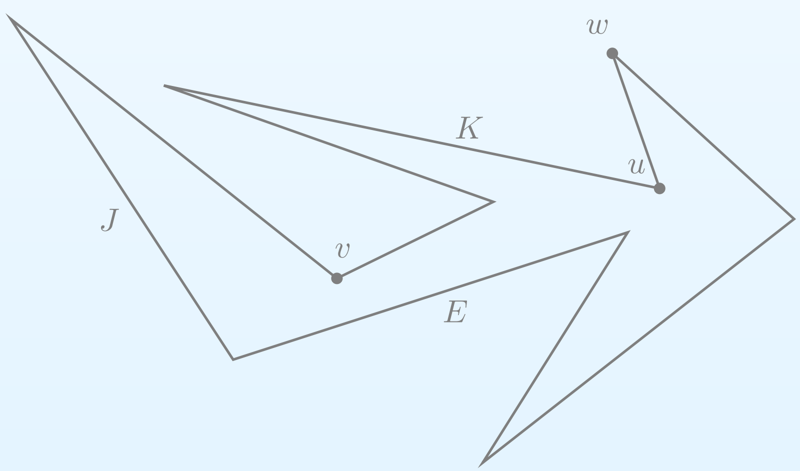


This is the *MAXIMUM VALUE VERTEX GUARD with PAINTING PLACEMENT (MVVGPP)* problem which has variations where the polygon might have holes, or the guards are realized as whole edges (edge guards).

Visibility Predicates

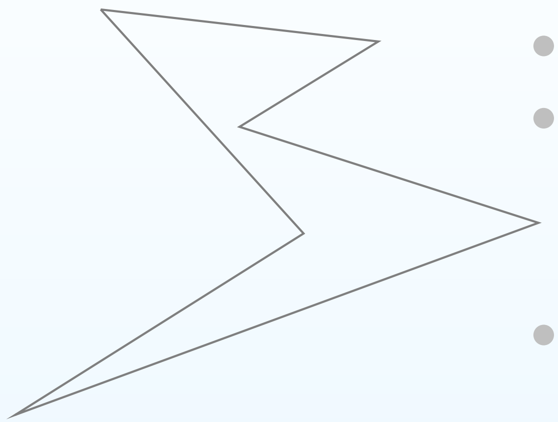
Let P be a polygon, $v, u \in P$ points in P and $E, J, K \subseteq P$ edges of P. We define the following visibility predicates:

- $sees(v, u) : \forall x \in \overline{vu} : x \in P$
- $oversees(v, E) : \forall x \in E : sees(v, x)$
- $oversees(E, J) : \forall x \in J : \exists y \in E : sees(x, y)$
- $watches(u, J) : \exists x \in J : sees(u, x)$
- $watches(E, K) : \exists x \in K : \exists y \in E : sees(x, y)$



Finest Visibility Segmentation

A Simple Construction



- Start with an arbitrary polygon.
- Find the Visibility Graph.
- Extend the visibility graph's edges, inside and up to the boundary of the polygon.
- $O(n^2)$ points are generated that include all vertices of the polygon.

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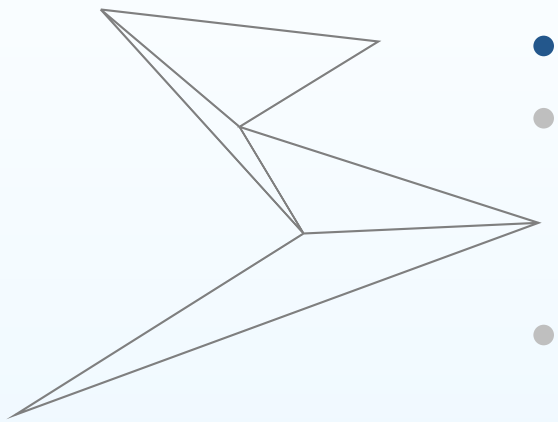

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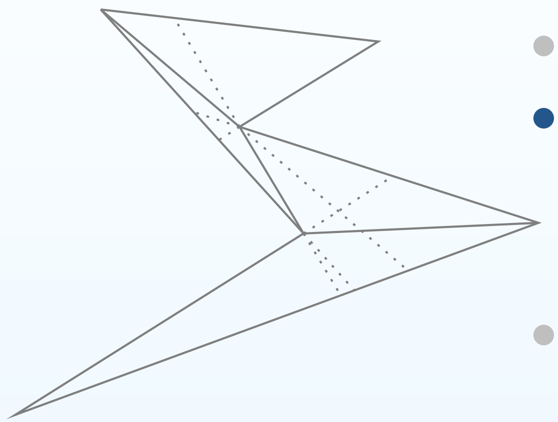

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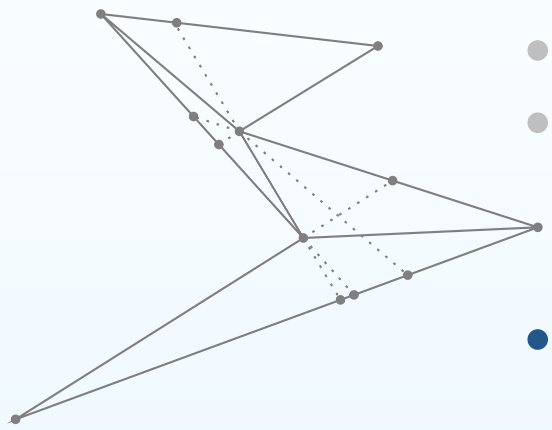

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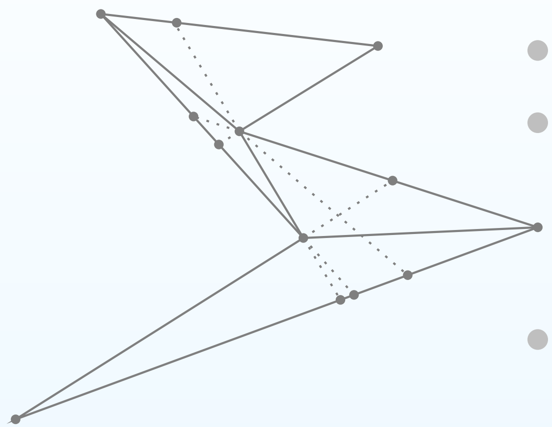

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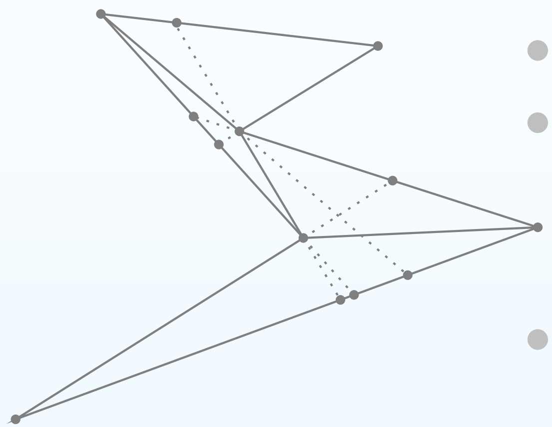

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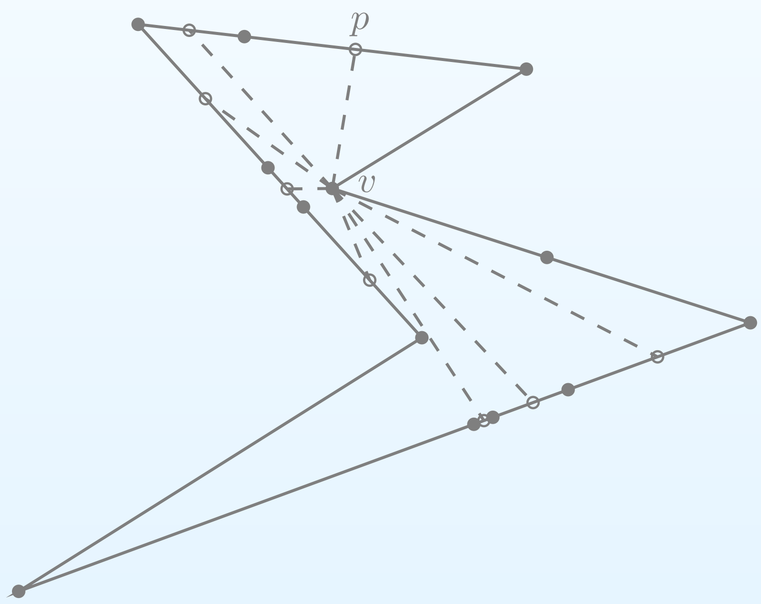
Theorem: Any vertex (edge) of P sees a FVS segment if and only if watches the FVS segment.

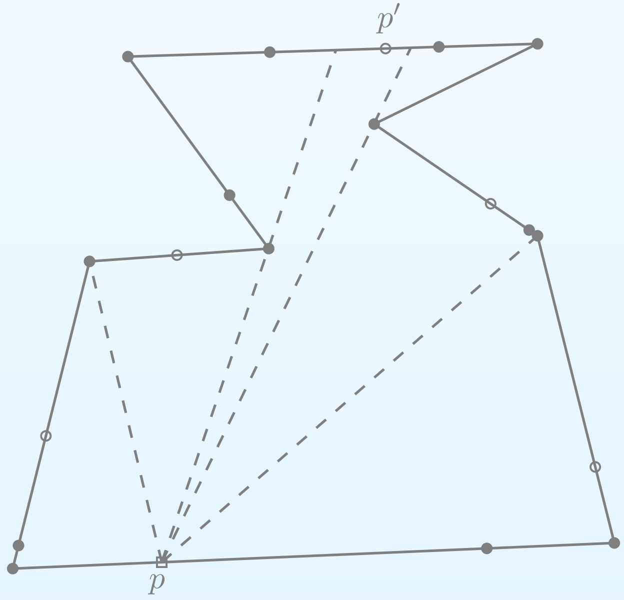
The *FVS(v)* set

In order to find the set $FVS(v)$ of all overseen FVS segments from a polygon vertex v, it suffices to pick an arbitrary point p in every FVS segment and check if vp is everywhere inside the polygon.



The *FVS(e)* set

In order to find the set $FVS(e)$ of all overseen FVS segments from a polygon edge e, it suffices to pick an arbitrary point p in every FVS segment and check if there exists a point $p' \in e$ such that pp' is everywhere inside the polygon.



The MVVGPP Problem

The Multiple Knapsack Problem

We are given a set of n items and m knapsacks such that each item i has a profit $p(i)$ and a size $s(i)$, and each knapsack j has a capacity $c(j)$. The goal is to find a subset of items of maximum profit such that they have a feasible packing in the knapsacks.

- The MK problem is a generalization of both the knapsack and bin packing problems and is strongly NP-hard.
- 2-approximation algorithm (Shmoys and Tardos, 1993)
- PTAS (Chekuri and Khanna, 2000)

The MVVGPP Problem

Given is a polygon P (art gallery), a set of ordered pairs (x, y) (paintings with length x and value y), and an integer $k > 0$.
The goal of the *MAXIMUM VALUE VERTEX GUARD with PAINTING PLACEMENT PROBLEM* is to place k vertex guards as well as place paintings on the boundary of P so that the total weight of the overseen paintings is maximum.

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Proof: The decision version of *MINIMUM VERTEX GUARD* problem for a polygon P reduces to the corresponding decision version of *MVVGPP* problem for the same polygon P: construct the FVS set and take a painting for each FVS segment that has length the length of the segment and value also the length of the segment. The total value that the *MVVGPP* problem asks for is the total sum of values of the segments.

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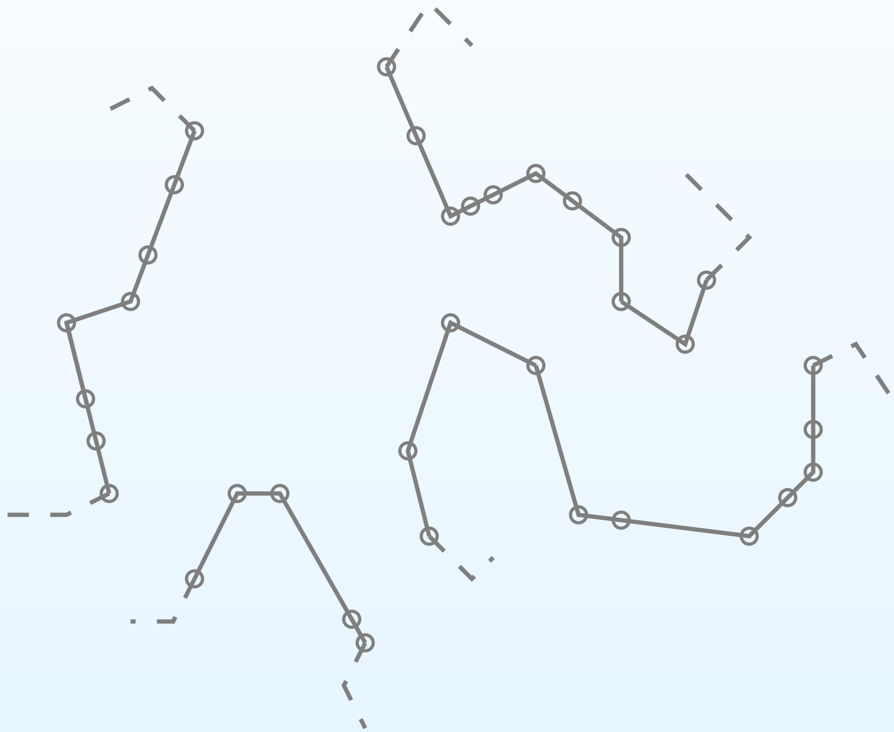
  - Update SOL: add the new FVS segments along with the fitted paintings.
- Return the total value of all the fitted paintings in SOL

Example



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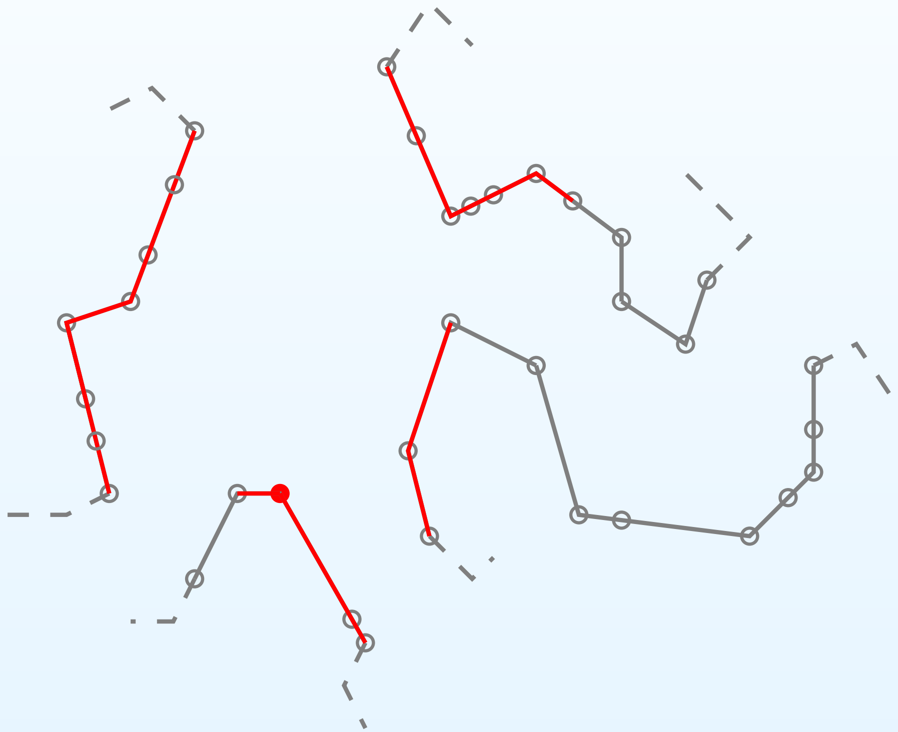

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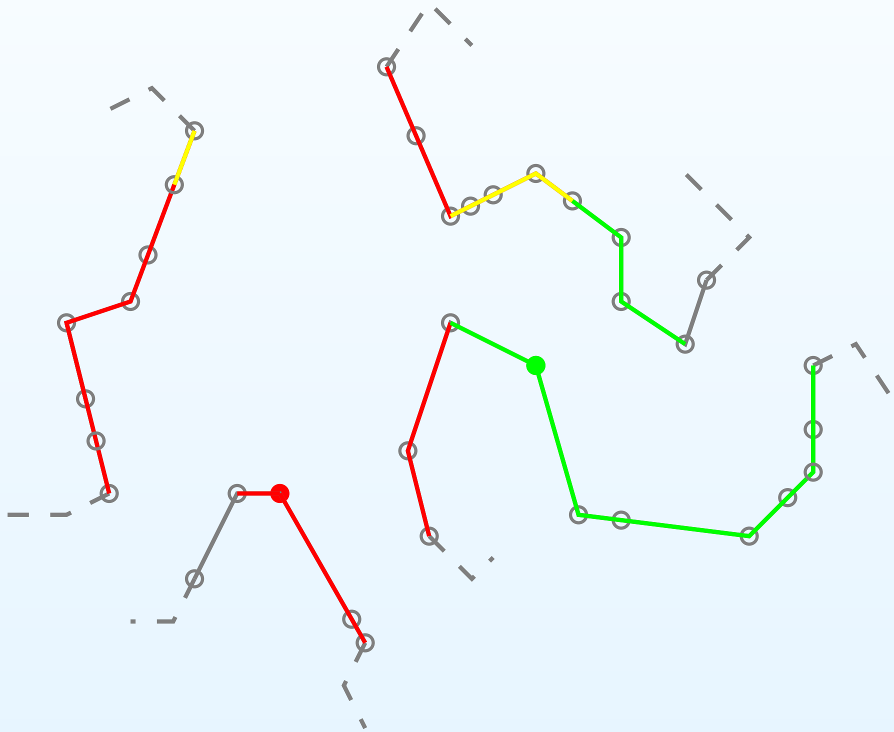

Remarks

- Let $W(SOL)$ the value returned from our algorithm and $W(OPT)$ the value of the collection of the set of paintings in an optimal solution. It holds: $W(SOL) > (1 - \frac{1}{e^a})\, W(OPT)$, where a is the approximation of the *MK* problem.

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- As $a \longrightarrow 1$, due to the existence of the PTAS for the MK problem, $1 - \frac{1}{e^a} \longrightarrow 0.633$

- Similar to *MVVGPP* problem the *MAXIMUM VALUE EDGE GUARD with PAINTING PLACEMENT* problem is NP-hard.
- The same algorithm applies except from the calculation of the $FVS(e)$ set.
- Our results are also applicable for polygons with holes.

Conclusion

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